

INVERSE SCATTERING APPROACH ON TOMOGRAPHY PROBLEM USING MULTI-FREQUENCY DATA

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ABSTRACT. An inverse scattering problem is formulated for reconstructing optical properties of biological tissues. A recursive linearization algorithm is used to solve the inverse scattering problem. We employed the idea of finite element boundary integral method and added suitable boundary conditions on the surface of the domain. The initial guess is obtained by Born approximation based on the fact of weak scattering. The reconstruction is then improved each time by an increment on wave number. Finite element method is used for the interior domain containing inhomogeneity. Nyström method is used for setting up the boundary conditions and jump conditions. Two numerical examples are presented.

1. INTRODUCTION

Photo-acoustic tomography has been shown great interest over the past decades and used to reconstruct optical property of biological tissue, such as the breast and the brain. It is a phenomenon in which the optical properties of the underlying medium is modified by absorbed radiation which in turn generates measurable acoustic waves. The acoustic signal is then collected to recover the property of the medium. Readers are referred to [1] for a description of the photo-acoustic

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effect and [2, 3, 4, 5, 6] for the development of the hybrid imaging modality, which combines the optical methods with the spatial resolution of ultrasound imaging.

The difficulty of acousto-optic imaging lies in various aspects of the reconstruction of absorbed radiation from the acoustic signal, such as limited data, spatially varying acoustic sound speed and the effects of acoustic wave attenuation. In [7], the authors assume the absorbed radiation known, then reconstructed the conductivity coefficient from the known absorbed radiation.

In this paper, we formulate the problem as an inverse scattering problem, which is to determine the conductivity property of the tissue from the measurements of electromagnetic field on the boundary of the medium, given the incident field.

Some related results can be found in [8] and [9], where the authors proposed a globally convergent numerical method for an inverse problem of recovering the coefficient from non overdetermined time dependent data with the single source location. The book [8] summarized results of its authors published in various journals in 2008-2011. Their algorithm was tested on both computationally simulated and experimental data.

Our approach follows the general idea of [10] and employs the recursive linearization algorithm from [11] and [12]. In their work the authors used fixed-frequency data, which contains multiple spatial frequency evanescent plane waves. In this paper multi-frequency data is used to recover the dielectric property of a dispersive medium, which depends on the wavenumber and has wider applications. A relevant convergence result can be found in [13], in which the authors proved the convergence of the algorithm along with an error estimate under some reasonable assumptions.

In two dimensional cases, the electromagnetic intensity satisfies the Helmholtz equation:

$$\Delta u + k_0^2(1 + q(k_0, x))u = 0, \quad (1.1)$$

where u is the total field; k_0 is the wavenumber in vacuum; $q(k_0, x)$ is the scatterer which has a compact support and $\epsilon(k_0, x) = 1 + q(k_0, x)$ is the dielectric permittivity in dispersive medium. We assumed that $q(k_0, x) = i\sigma(x)/k_0$, where $\sigma(x)$ is the conductivity of the medium. In the following, we assume that the material is nonmagnetic, i.e., $\mu_0 = 1$.

The scatterer is illuminated by a one-parameter family of plane waves

$$u^i = \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_1), \quad (1.2)$$

where $\mathbf{d}_1 = (\cos \theta, \sin \theta)$, $\theta \in [0, 2\pi]$. Evidently, such incident waves satisfy the homogeneous equation

$$\Delta u^i + k_0^2 u^i = 0. \quad (1.3)$$

The total electric field u consists of the incident field u^i and the scattered field u^s :

$$u = u^i + u^s. \quad (1.4)$$

It follows from the equations (1.1) and (1.2) that the scattered field satisfies

$$\Delta u^s + k_0^2(1 + q)u^s = -k_0^2 q u^i. \quad (1.5)$$

In free space, the scattered field is required to satisfy the following Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - ik_0 u^s \right) = 0, \quad r = |x|, \quad (1.6)$$

uniformly along all directions $x/|x|$. In practice, it is convenient to reduce the problem to a bounded domain. For the sake of simplicity, we employ the first order absorbing boundary condition [14] on the surface of the medium:

$$\frac{\partial u^s}{\partial n} - ik_0 u^s = 0. \quad (1.7)$$

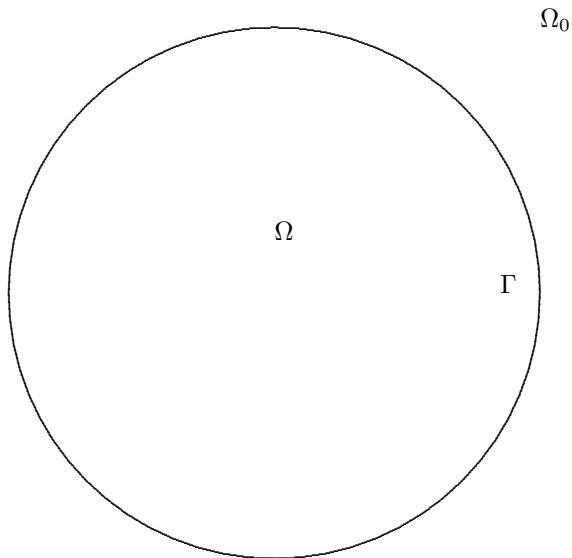


FIGURE 1. Geometry of the inverse scattering problem

Given the incident field u^i , the direct problem is to determine the scattered field u^s for the known scatterer $q(k_0, x)$. Using the Lax-Milgram lemma and the Fredholm alternative, the direct problem is shown in [11] to have a unique solution for all $k_0 > 0$. An energy estimate for the scattered field is given in this paper, which provides a criterion for the weak scattering. Furthermore, properties on the continuity and the Fréchet differentiability of the nonlinear scattering map are examined. For the regularity analysis of the scattering map in an open domain, the reader is referred to [15], [16] and [17]. The inverse medium scattering problem is to determine the scatterer $q(k_0, x)$ from the measurements on the surface of the medium, $u^s|_{\Gamma}$, given the incident field u^i . Two major difficulties for solving the inverse problem by optimization methods are the ill-posedness and the presence of many local minima. In this paper we developed a continuation method based on the approach introduced in [11]. The algorithm requires multi-frequency scattering data. Using an initial guess from the Born approximation, each update is obtained via recursive linearization on the wavenumber k_0 by solving one forward problem and one adjoint problem of the Helmholtz equations.

The plan of this paper is as follows. The analysis of the variational problem for direct scattering is presented in section 2. The Fréchet differentiability of the

scattering map is also given. In section 3.1, an initial guess of the reconstruction from the Born approximation is derived in the case of weak scattering. Section 3.2 is devoted to numerical study of a regularized iterative linearization algorithm. In section 4, we discuss the numerical implementation of the forward scattering problem and the recursive linearization algorithm. Numerical examples are presented in section 5.

2. ANALYSIS OF THE SCATTERING MAP

In this section, the direct scattering problem is studied to provide some criterion for the weak scattering, which plays an important role in the inversion method. The Fréchet differentiability of the scattering map for the problem (1.5), (1.7) is examined.

To obtain the variational form of our boundary value problem, we multiply (1.5) by a test function ψ , and integrate:

$$\int_{\Omega} (\Delta u^s \psi + k_0(1+q)u^s \psi) dx = \int_{\Omega} -k_0^2 q u^i \psi dx.$$

Using Green's First Identity, we get

$$-\int_{\Omega} \nabla u^s \nabla \psi dx + \int_{\Gamma} \frac{\partial u^s}{\partial n} \psi ds + \int_{\Omega} k_0(1+q)u^s \psi dx = \int_{\Omega} -k_0^2 q u^i \psi dx.$$

By (1.7), it is equivalent to

$$-\int_{\Omega} \nabla u^s \nabla \psi dx + \int_{\Gamma} i k_0 u^s \psi ds + \int_{\Omega} k_0(1+q)u^s \psi dx = \int_{\Omega} -k_0^2 q u^i \psi dx.$$

Therefore, we introduce the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$

$$a(\phi, \psi) = (\nabla \phi, \nabla \psi) - k_0^2((1+q)\phi, \psi) - i k_0 \langle \phi, \psi \rangle, \quad (2.1)$$

and the linear functional on $H^1(\Omega)$

$$b(\psi) = (k_0^2 q u^i, \psi). \quad (2.2)$$

Here, we have used the standard inner products

$$(\phi, \psi) = \int_{\Omega} \phi \cdot \bar{\psi} dx \quad \text{and} \quad \langle \phi, \psi \rangle = \int_{\Gamma} \phi \cdot \bar{\psi} ds, \quad (2.3)$$

where the overline denotes the complex conjugate.

Then, we have the weak form of the boundary value problem (1.5) and (1.7): find $u^s \in H^1(\Omega)$ such that

$$a(u^s, \xi) = b(\xi), \quad \forall \xi \in H^1(\Omega). \quad (2.4)$$

Throughout the paper, the constant C stands for a positive generic constant whose value may change step by step, but should always be clear from the contexts.

For a given scatterer q and an incident field u^i , suppose a solution of the problem (1.5) and (1.7) or the variational problem (2.4) $u^s \in H^1(\Omega)$ exists, we define the map $S(q, u^i)$ by $u^s = S(q, u^i)$. It is easily seen that the map $S(q, u^i)$ is linear with respect to u^i but is nonlinear with respect to q . Hence, we may denote $S(q, u^i)$ by $S(q)u^i$. In the following, we will show the existence of the solution by Fredholm alternative.

Concerning the map $S(q)$, a continuity result for the map $S(q)$ is presented in Lemma 2.3.

Lemma 2.1. *Given the scatterer $q \in L^\infty(\Omega)$, the direct scattering problem (1.5) and (1.7) has at most one solution.*

Proof. It is sufficient to show that $u^s = 0$ if $u^i = 0$. We begin by noting from (1.5) that

$$\begin{aligned} & \Im \left\{ \int_{\Omega} (u^s \Delta \bar{u}^s - \bar{u}^s \Delta u^s) dV \right\} \\ &= \Im \left\{ \int_{\Omega} [u^s (-k_0^2 (1 + \bar{q}) \bar{u}^s) - \bar{u}^s (-k_0^2 (1 + q) u^s)] dV \right\} \\ &= \Im \left\{ \int_{\Omega} |u^s|^2 (-k_0^2 (1 + \bar{q}) + k_0^2 (1 + q)) dV \right\} \\ &= \int_{\Omega} |u^s|^2 k_0^2 (1 + 2\Im(q)) dV \geq 0. \end{aligned}$$

On the other hand by the absorbing boundary condition (1.7)

$$\begin{aligned}
& \Im \left\{ \int_{\Omega} (u^s \Delta \bar{u}^s - \bar{u}^s \Delta u^s) dV \right\} \\
&= \Im \left\{ \oint_{\Gamma} \left(u^s \frac{\partial \bar{u}^s}{\partial n} - \bar{u}^s \frac{\partial u^s}{\partial n} \right) dS \right\} \\
&= \Im \left\{ \oint_{\Gamma} (-ik_0 u^s \bar{u}^s - ik_0 \bar{u}^s u^s) dS \right\} \\
&= -2k_0 \oint_{\Gamma} \|u^s\|^2 dS.
\end{aligned}$$

Hence we have $u^s = 0$ on Γ and $\frac{\partial u^s}{\partial n} = 0$ on Γ , which implies $u^s = 0$ in Ω by a unique continuation result for Helmholtz equation [16]. \square

Lemma 2.2. *If the wavenumber k_0 is sufficiently small, the variational problem (2.4) admits a unique weak solution in $H^1(\Omega)$ and $S(q)$ is a bounded linear map from $L^2(\Omega)$ to $H^1(\Omega)$. Furthermore, there is a constant C dependent of Ω , such that*

$$\|S(q)u^i\|_{H^1(\Omega)} \leq Ck_0^2 \|q\|_{L^\infty(\Omega)} \|u^i\|_{L^2(\Omega)}. \quad (2.5)$$

Proof. Decompose the bilinear form a into $a = a_1 + k_0^2 a_2$, where

$$a_1(u^s, \xi) = (\nabla u^s, \nabla \xi) - ik_0 \langle u^s, \xi \rangle, \quad (2.6)$$

$$a_2(u^s, \xi) = -((1+q)u^s, \xi). \quad (2.7)$$

We conclude that a_1 is coercive from

$$\begin{aligned}
|a_1(u^s, u^s)| &= \| \nabla u^s \|_{L^2(\Omega)}^2 - ik_0 \|u^s\|_{L^2(\Gamma)}^2 \\
&= \sqrt{\| \nabla u^s \|_{L^2(\Omega)}^4 + k_0^2 \|u^s\|_{L^2(\Gamma)}^4} \\
&\geq C \|u^s\|_{H^1(\Omega)}^2,
\end{aligned}$$

where the last inequality may be obtained by Poincaré inequality. Next, we prove the compactness of a_2 . Define an operator $\mathcal{A} : L^2(\Omega) \rightarrow H^1(\Omega)$ by

$$a_1(\mathcal{A}u^s, \xi) = a_2(u^s, \xi), \quad \forall \xi \in H^1(\Omega), \quad (2.8)$$

which gives

$$(\nabla \mathcal{A}u^s, \nabla \xi) - ik_0 \langle \mathcal{A}u^s, \xi \rangle = -((1+q)u^s, \xi), \quad \forall \xi \in H^1(\Omega).$$

Using the Lax–Milgram Lemma, it follows that

$$\|\mathcal{A}u^s\|_{H^1(\Omega)} \leq C \|u^s\|_{L^2(\Omega)},$$

where the constant C is independent of k_0 . Thus \mathcal{A} is bounded from $L^2(\Omega)$ to $H^1(\Omega)$ and $H^1(\Omega)$ is compactly imbedded into $L^2(\Omega)$. Hence $\mathcal{A} : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact operator.

Define a function $\phi \in L^2(\Omega)$ by requiring $\phi \in H^1(\Omega)$ and satisfying

$$a_1(\phi, \xi) = b(\xi), \quad \forall \xi \in H^1(\Omega). \quad (2.9)$$

It follows from the Lax–Milgram Lemma again that

$$\|\phi\|_{H^1(\Omega)} \leq C k_0^2 \|q\|_{L^\infty(\Omega)} \|u^i\|_{L^2(\Omega)}. \quad (2.10)$$

Using the operator \mathcal{A} , we can see that problem (2.4) is equivalent to find $u^s \in L^2(\Omega)$ such that

$$(\mathcal{I} + k_0^2 \mathcal{A})u^s = \phi. \quad (2.11)$$

When the wavenumber k_0 is small enough, the operator $\mathcal{I} + k_0^2 \mathcal{A}$ has a uniformly bounded inverse. We then have the estimate

$$\|u^s\|_{L^2(\Omega)} \leq C \|\phi\|_{L^2(\Omega)}, \quad (2.12)$$

where the constant C is independent of k_0 . Rearranging (2.11), we have $u^s = \phi - k_0^2 \mathcal{A}u^s$, so $u^s \in H^1(\Omega)$ and, by the estimate (2.9) for the operator \mathcal{A} , we have

$$\|u^s\|_{H^1(\Omega)} \leq \|\phi\|_{H^1(\Omega)} + Ck_0^2 \|u^s\|_{L^2(\Omega)}.$$

The proof is complete by combining the estimates (2.12) and (2.10) and observing that $u^s = S(q)u^i$. \square

For a general wavenumber $k_0 > 0$, from the equation (2.11), the existence follows from the Fredholm alternative and the uniqueness result.

Remark 2.1. It follows from the explicit form of the incident field (1.2) and the estimate (2.5) that

$$\|u^s\|_{H^1(\Omega)} \leq Ck_0^2 |\Omega|^{\frac{1}{2}} \|q\|_{L^\infty(\Omega)}, \quad (2.13)$$

where the constant C depends on Ω .

Lemma 2.3. *Assume that $q_1, q_2 \in L^\infty(\Omega)$. Then*

$$\|S(q_1)u^i - S(q_2)u^i\|_{H^1(\Omega)} \leq C \|q_1 - q_2\|_{L^\infty(\Omega)} \|u^i\|_{L^2(\Omega)}, \quad (2.14)$$

where the constant C depends on k_0, Ω , and $\|q_2\|_{L^\infty(\Omega)}$.

Proof. Let $u_1^s = S(q_1)u^i$ and $u_2^s = S(q_2)u^i$. It follows that for $j = 1, 2$

$$\Delta u_j^s + k_0^2(1 + q_j)u_j^s = -k_0^2 q_j u^i.$$

By setting $w = u_1^s - u_2^s$, we have

$$\Delta w + k_0^2(1 + q_1)w = -k_0^2(q_1 - q_2)(u^i + u_2^s).$$

The function w also satisfies the boundary condition (1.7).

We repeat the procedure in the proof of Lemma 2.2

$$\|w\|_{H^1(\Omega)} \leq C\|q_1 - q_2\|_{L^\infty(\Omega)}\|u^i + u_2^s\|_{L^2(\Omega)}.$$

Using Lemma 2.2 again for u_2^s yields

$$\|u_2^s\|_{H^1(\Omega)} \leq C\|q_2\|_{L^\infty(\Omega)}\|u^i\|_{L^2(\Omega)},$$

which gives

$$\|S(q_1)u^i - S(q_2)u^i\|_{H^1(\Omega)} \leq C\|q_1 - q_2\|_{L^\infty(\Omega)}\|u^i\|_{L^2(\Omega)},$$

where the constant C depends on Ω, k_0 , and $\|q_2\|_{L^\infty(\Omega)}$. \square

Let γ be the restriction (trace) operator to the boundary Γ . By the trace theorem, γ is a bounded linear operator from $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$. We can now define the scattering map $M(q) = \gamma S(q)$.

Next, consider the Fréchet differentiability of the scattering map. Recall the map $S(q)$ is nonlinear with respect to q . Formally, by using the first order perturbation theory, we obtain the linearized scattering problem of (1.5), (1.7) with respect to a reference scatterer q ,

$$\Delta v + k_0^2(1 + q)v = -k_0^2\delta q(u^i + u^s), \quad (2.15)$$

$$\frac{\partial v}{\partial n} - ik_0v = 0,$$

where $u^s = S(q)u^i$.

Define the formal linearization $T(q)(\delta q)$ of the map $S(q)u^i$ by $v = T(q)(\delta q, u^i)$, where v is the solution of the problem (2.15). The following is a boundedness result for the map $T(q)$. A proof may be given by following step by step the proofs of Lemma 2.2. Hence we omit it here.

Lemma 2.4. *Assume that $q, \delta q \in L^\infty(\Omega)$ and u^i is the incident field. Then $v = T(q)(\delta q, u^i) \in H^1(\Omega)$ with the estimate*

$$\|T(q)(\delta q, u^i)\|_{H^1(\Omega)} \leq C \|\delta q\|_{L^\infty(\Omega)} \|u^i\|_{L^2(\Omega)}, \quad (2.16)$$

where the constant C depends on k_0, Ω , and $\|q\|_{L^\infty(\Omega)}$.

The next lemma is concerned with the continuity property of the map.

Lemma 2.5. *For any $q_1, q_2 \in L^\infty(\Omega)$ and an incident field u^i , the following estimate holds*

$$\|T(q_1)(\delta q, u^i) - T(q_2)(\delta q, u^i)\|_{H^1(\Omega)} \leq C \|q_1 - q_2\|_{L^\infty(\Omega)} \|\delta q\|_{L^\infty(\Omega)} \|u^i\|_{L^2(\Omega)}, \quad (2.17)$$

where the constant C depends on k_0, Ω , and $\|q_2\|_{L^\infty(\Omega)}$.

Proof. Let $v_j = T(q_j)(\delta q, u^i)$, for $j = 1, 2$. It is easy to see that

$$\Delta(v_1 - v_2) + k_0^2(1 + q_1)(v_1 - v_2) = -k_0^2 \delta q(u_1^s - u_2^s) - k_0^2(q_1 - q_2)v_2,$$

where $u_j^s = S(q_j)u^i$.

Similar to the proof of Lemma 2.2, we get

$$\|v_1 - v_2\|_{H^1(\Omega)} \leq C(\|\delta q\|_{L^\infty(\Omega)} \|u_1^s - u_2^s\|_{H^1(\Omega)} + \|q_1 - q_2\|_{L^\infty(\Omega)} \|v_2\|_{H^1(\Omega)}).$$

From Lemma 2.3 and Lemma 2.4, we obtain

$$\|v_1 - v_2\|_{H^1(\Omega)} \leq C \|q_1 - q_2\|_{L^\infty(\Omega)} \|\delta q\|_{L^\infty(\Omega)} \|u^i\|_{L^2(\Omega)},$$

which completes the proof. \square

The following result concerns the differentiability property of $S(q)$.

Lemma 2.6. *Assume that $q, \delta q \in L^\infty(\Omega)$. Then there is a constant C dependent of k_0, Ω , and $\|q\|_{L^\infty(\Omega)}$, for which the following estimate holds*

$$\|S(q + \delta q)u^i - S(q)u^i - T(q)(\delta q, u^i)\|_{H^1(\Omega)} \leq C\|\delta q\|_{L^\infty(\Omega)}^2 \|u^i\|_{L^2(\Omega)}. \quad (2.18)$$

Proof. By setting $u_1^s = S(q)u^i$, $u_2^s = S(q + \delta q)u^i$, and $v = T(q)(\delta q, u^i)$, we have

$$\Delta u_1^s + k_0^2(1 + q)u_1^s = -k_0^2 q u^i, \quad (2.19)$$

$$\Delta u_2^s + k_0^2(1 + q + \delta q)u_2^s = -k_0^2(q + \delta q)u^i, \quad (2.20)$$

$$\Delta v + k_0^2(1 + q)v = -k_0^2 \delta q u_1^s - k_0^2 \delta q u^i. \quad (2.21)$$

In addition, u_1^s, u_2^s , and v satisfy the boundary condition (1.7).

Denote $U = u_2^s - u_1^s - v$. Then

$$\Delta U + k_0^2(1 + q)U = -k_0^2 \delta q (u_2^s - u_1^s). \quad (2.22)$$

Similar arguments as in the proof of Lemma 2.2 give

$$\|U\|_{H^1(\Omega)} \leq C\|\delta q\|_{L^\infty(\Omega)} \|u_2^s - u_1^s\|_{H^1(\Omega)}.$$

From Lemma 2.2, we obtain further that

$$\|U\|_{H^1(\Omega)} \leq C\|\delta q\|_{L^\infty(\Omega)}^2 \|u^i\|_{L^2(\Omega)}.$$

□

Finally, by combining the above lemmas, we arrive at

Theorem 2.1. *The scattering map $M(q)$ is Fréchet differentiable with respect to q and its Fréchet derivative is*

$$DM(q) = \gamma T(q). \quad (2.23)$$

3. INVERSE MEDIUM SCATTERING

In this section, a regularized recursive linearization method for solving the inverse scattering problem of the Helmholtz equation in two dimensions is proposed. The algorithm requires multi-frequency Dirichlet and Neumann scattering data, and the recursive linearization is obtained by a continuation method on the wavenumber k_0 . It first solves a linear equation (Born approximation) at the lowest k_0 , which gives the initial guess of $q(k_0, x)$. Updates are subsequently obtained by using a sequence of increasing wavenumbers. For each iteration, one forward and one adjoint equation are solved.

Let Ω be the circle that contains the underlying medium; let $\Gamma = \partial\Omega$ be the surface; let $\Omega_0 = \mathbb{R}^2/\bar{\Omega}$. The inverse problem can be stated as follows. Given $u^s(x; k)$ for all $x \in \Gamma$, and all $k > 0$, find the function $q(x)$, $x \in \Omega$, assuming that conditions (3.1)-(3.5) hold:

$$\Delta u + k_0^2(1 + q)u = 0 \quad \text{in } \Omega, \quad (3.1)$$

$$\Delta u^0 + k_0^2 u^0 = 0 \quad \text{in } \Omega_0, \quad (3.2)$$

$$u = u^0 = u^s + u^i \quad \text{on } \Gamma, \quad (3.3)$$

$$\frac{\partial u}{\partial n} = \frac{\partial u^s}{\partial n} + \frac{\partial u^i}{\partial n} \quad \text{on } \Gamma, \quad (3.4)$$

$$\frac{\partial u^s}{\partial n} - ik_0 u^s = 0 \quad \text{on } \Gamma, \quad (3.5)$$

where (3.3) and (3.4) are the jump conditions on the surface of the medium.

3.1. Born Approximation. Define a test function $\hat{u} = \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_2)$, $\mathbf{d}_2 = (\cos \theta, \sin \theta)$, $\theta \in [0, 2\pi]$. Hence \hat{u} satisfies:

$$\Delta \hat{u} + k_0^2 \hat{u} = 0 \quad \text{in } \Omega. \quad (3.6)$$

Multiplying the equation (3.1) by \hat{u} , and integrating over Ω on both sides, we have

$$\int_{\Omega} \hat{u} \Delta u dx + k_0^2 \int_{\Omega} (1 + q) \hat{u} u dx = 0.$$

Integration by parts yields

$$\int_{\Omega} \Delta \hat{u} u dx + \int_{\Omega} k_0^2 (1 + q) u \hat{u} dx + \int_{\Gamma} (\hat{u} \frac{\partial u}{\partial n} - u \frac{\partial \hat{u}}{\partial n}) ds = 0.$$

We have by noting (3.6) and the jump conditions (3.3) and (3.4) that

$$\int_{\Omega} k_0^2 q u \hat{u} dx = \int_{\Gamma} (u^s \frac{\partial \hat{u}}{\partial n} - \hat{u} \frac{\partial u^s}{\partial n}) ds + \int_{\Gamma} (u^i \frac{\partial \hat{u}}{\partial n} - \hat{u} \frac{\partial u^i}{\partial n}) ds,$$

where we take into account that q has compact support in Ω . Using the special form of the incident wave and the test function, we then get

$$\begin{aligned} & \int_{\Omega} k_0^2 q u \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_1) dx \\ &= \int_{\Gamma} (ik_0 \mathbf{n} \cdot \mathbf{d}_1 \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_1) u^s - \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_1) \frac{\partial u^s}{\partial n}) ds \\ &+ \int_{\Gamma} \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_2 + ik_0 \mathbf{x} \cdot \mathbf{d}_1) (ik_0 \mathbf{n} \cdot \mathbf{d}_1 - ik_0 \mathbf{n} \cdot \mathbf{d}_2) ds. \end{aligned} \quad (3.7)$$

From Lemma 2.2 and Remark 2.1, for a small wavenumber, the scattered field is weak comparing to the incident field, so $u = u^s + u^i \approx u^i$. We drop the nonlinear term of (3.7) and obtain the linearized integral equation

$$\begin{aligned} & \int_{\Omega} k_0^2 q_0(x) \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_2 + ik_0 \mathbf{x} \cdot \mathbf{d}_1) dx \\ &= \int_{\Gamma} (ik_0 \mathbf{n} \cdot \mathbf{d}_1 \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_1) u^s - \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_1) \frac{\partial u^s}{\partial n}) ds \\ &+ \int_{\Gamma} \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_2 + ik_0 \mathbf{x} \cdot \mathbf{d}_1) (ik_0 \mathbf{n} \cdot \mathbf{d}_1 - ik_0 \mathbf{n} \cdot \mathbf{d}_2) ds, \end{aligned} \quad (3.8)$$

which is the Born approximation.

Since the scatterer $q_0(k_0, x)$ has a compact support, we use the notation

$$\hat{q}_0(\xi) = \int_{\Omega} q_0(x) \exp(ik_0 \mathbf{x} \cdot \mathbf{d}_2 + ik_0 \mathbf{x} \cdot \mathbf{d}_1) dx, \quad (3.9)$$

where $\hat{q}_0(\xi)$ is the Fourier transform of $q_0(x)$ with $\xi = k_0(\mathbf{d}_1 + \mathbf{d}_2)$. Choose

$$\mathbf{d}_j = (\cos \theta_j, \sin \theta_j), \quad j = 1, 2, \quad (3.10)$$

where θ_j are spherical angles. It is obvious that the domain $[0, 2\pi]$ of $\theta_j, j = 1, 2$, corresponds to the ball $\{\xi \in \mathbb{R}^2 : |\xi| \leq 2k_0\}$. Thus, the Fourier modes of $\hat{q}_0(\xi)$ in the ball $\{\xi : |\xi| \leq 2k_0\}$ can be determined. The scattering data with the higher wavenumber must be used in order to recover more modes of the true scatterer.

The integral equation (3.8) can be written as the operator form

$$A(k_0, \theta; x)q(x) = f(k_0, \theta). \quad (3.11)$$

Thus

$$q_0(k_0, x) = (A * A + \alpha I)^{-1}f, \quad (3.12)$$

where α is a relaxation parameter. q_0 is used as the starting point of the following recursive linearization algorithm.

3.2. Recursive Linearization. As discussed in the previous section, when the wavenumber is small, the Born approximation allows a reconstruction of those Fourier modes less than or equal to $2k_0$ for the function $q(x)$. We now describe a procedure that recursively determines q_{k_0} at $k_0 = k_j$ for $j = 1, 2, \dots$ with the increasing wavenumbers. Suppose now that the scatterer $q_{\tilde{k}}$ has been recovered at some wavenumber \tilde{k} , and that the wavenumber k is slightly larger than \tilde{k} . We wish to determine q_k , or equivalently, to determine the perturbation

$$\delta q = q_k - q_{\tilde{k}}. \quad (3.13)$$

For the reconstructed scatterer $q_{\tilde{k}}$, we solve at the wavenumber k the forward scattering problem

$$\begin{aligned}
\Delta \tilde{u} + k^2(1 + q_{\tilde{k}}(k, x))\tilde{u} &= 0 && \text{in } \Omega, \\
\Delta \tilde{u}^0 + k^2\tilde{u}^0 &= 0 && \text{in } \Omega_0, \\
\tilde{u} = \tilde{u}^0 = \tilde{u}^s + u^i &&& \text{on } \Gamma, \\
\frac{\partial \tilde{u}}{\partial n} = \frac{\partial \tilde{u}^s}{\partial n} + \frac{\partial u^i}{\partial n} &&& \text{on } \Gamma, \\
\frac{\partial \tilde{u}^s}{\partial n} - ik\tilde{u}^s &= 0 && \text{on } \Gamma.
\end{aligned} \tag{3.14}$$

For the scatterer q_k , we have

$$\begin{aligned}
\Delta u + k^2(1 + q_k(k, x))u &= 0 && \text{in } \Omega, \\
\Delta u^0 + k^2u^0 &= 0 && \text{in } \Omega_0, \\
u = u^0 = u^s + u^i &&& \text{on } \Gamma, \\
\frac{\partial u}{\partial n} = \frac{\partial u^s}{\partial n} + \frac{\partial u^i}{\partial n} &&& \text{on } \Gamma, \\
\frac{\partial u^s}{\partial n} - ik u^s &= 0 && \text{on } \Gamma.
\end{aligned} \tag{3.15}$$

Subtracting (3.14) from (3.15) and omitting the second-order smallness in δq and in $\delta u = u - \tilde{u}$, we obtain

$$\begin{aligned}
\Delta \delta u + k^2(1 + q_{\tilde{k}}(k, x))\delta u &= -k^2\delta q\tilde{u} && \text{in } \Omega, \\
\Delta \delta u^0 + k^2\delta u^0 &= 0 && \text{in } \Omega_0, \\
\delta u = \delta u^0 = \delta u^s &&& \text{on } \Gamma, \\
\frac{\partial \delta u}{\partial n} = \frac{\partial \delta u^s}{\partial n} &&& \text{on } \Gamma, \\
\frac{\partial \delta u^s}{\partial n} - ik\delta u^s &= 0 && \text{on } \Gamma.
\end{aligned} \tag{3.16}$$

For the scatterer q_k , we define the scattering map

$$M(q_k) = u^0|_{\Gamma}, \quad (3.17)$$

where u^0 is the total field data corresponding to the incident wave u^i . Let $DM(q_k)$ be the Fréchet derivative of $M(q_k)$ and denote the residual operator by

$$R(q_{\bar{k}}) = u^0|_{\Gamma} - \tilde{u}^0|_{\Gamma}. \quad (3.18)$$

It follows from (2.23) that

$$DM(q_{\bar{k}})\delta q = R(q_{\bar{k}}). \quad (3.19)$$

In order to reduce the computation cost and instability, we consider the one-step Landweber iteration of (3.19), which has the form

$$\delta q = \beta DM^*(q_{\bar{k}})R(q_{\bar{k}}), \quad (3.20)$$

where β is a relaxation parameter and $DM^*(q_{\bar{k}})$ is the adjoint operator of $DM(q_{\bar{k}})$.

In order to compute the correction δq , we need some efficient way to compute $DM^*(q_{\bar{k}})R(q_{\bar{k}})$, which is given by the following theorem.

Theorem 3.1. *Given residual $R(q_{\bar{k}})$, the adjoint Fréchet derivative $DM^*(q_{\bar{k}})$ satisfies*

$$[DM^*(q_{\bar{k}})R(q_{\bar{k}})](x) = \bar{\tilde{u}}(x) \cdot \psi, \quad (3.21)$$

where \tilde{u} is the solution of (3.14), ψ is the solution of the adjoint problem of (3.16).

Proof. Let \tilde{u} be the solution of (3.14), and δu be the solution of (3.16).

Multiplying the first equation in (3.16) with the complex conjugate of a test function ψ and integrating over Ω on both sides, we obtain

$$\int_{\Omega} \Delta \delta u \bar{\psi} dx + \int_{\Omega} k^2(1 + q_{\bar{k}}) \delta u \bar{\psi} dx = \int_{\Omega} -k^2 \delta q \tilde{u} \bar{\psi} dx.$$

Green's second identity yields

$$\int_{\Omega} \delta u \Delta \bar{\psi} dx + \int_{\Gamma} (\bar{\psi} \frac{\partial \delta u}{\partial n} - \delta u \frac{\partial \bar{\psi}}{\partial n}) ds + \int_{\Omega} k^2 (1 + q_{\bar{k}}) \delta u \bar{\psi} dx = -k^2 \int_{\Omega} \delta q \tilde{u} \bar{\psi} dx.$$

We now define the adjoint problem

$$\begin{aligned} \Delta \psi + \bar{k}^2 (1 + \bar{q}_{\bar{k}}(k, x)) \psi &= 0 & \text{in } \Omega, \\ \Delta \psi + k^2 \psi &= 0 & \text{in } \Omega_0, \\ \frac{\partial \psi}{\partial n} + i \bar{k} \psi &= (u^0 - \tilde{u}^0) \bar{k}^2 & \text{on } \Gamma. \end{aligned} \tag{3.22}$$

Since the existence and uniqueness of the weak solution for the adjoint problem may be established by following the same proof of Lemma 2.2, we omit the proof here.

The above equation is then reduced to

$$\int_{\Gamma} (\bar{\psi} \frac{\partial \delta u}{\partial n} - \delta u \frac{\partial \bar{\psi}}{\partial n}) ds = -k^2 \int_{\Omega} \delta q \tilde{u} \bar{\psi} dx.$$

It follows from boundary conditions of (3.16) and the adjoint problem that

$$\int_{\Gamma} \overline{(u^0 - \tilde{u}^0)} k^2 \delta u ds = k^2 \int_{\Omega} \delta q \tilde{u} \bar{\psi} dx,$$

Extending the integration domain to Ω and using the operators defined above, we obtain

$$\int_{\Omega} DM(q_{\bar{k}}) \delta q \overline{R(q_{\bar{k}})} dx = \int_{\Omega} \delta q \tilde{u} \bar{\psi} dx.$$

We know from the adjoint operator $DM^*(q_{\bar{k}})$ that

$$\int_{\Gamma} \delta q \overline{DM^*(q_{\bar{k}}) R(q_{\bar{k}})} ds = \int_{\Omega} \delta q \tilde{u} \bar{\psi} dx.$$

Since it holds for any δq , we have

$$\overline{DM^*(q_{\bar{k}}) R(q_{\bar{k}})} = \tilde{u} \bar{\psi}.$$

Taking the complex conjugate of the above equation yields the result. \square

Using this theorem, we can rewrite (3.20) as

$$\delta q = \beta \bar{u} \psi. \quad (3.23)$$

So for each incident wave and each wavenumber k_0 , we have to solve one forward problem (3.14) along with one adjoint problem (3.22). Since the adjoint problem has a similar variational form as the forward problem, essentially, we need to compute two forward problems at each sweep. Once δq is determined, $q_{\bar{k}}$ is updated by $q_{\bar{k}} + \delta q$.

4. IMPLEMENTATION

In this section, we discuss the numerical solution of the forward scattering problem and the computational issues of the recursive linearization algorithm.

The scattering data are obtained by numerical solution of the forward scattering problem. To implement the algorithm numerically, we employ Nyström's method in the exterior region Ω_0 and add some suitable boundary conditions on Γ . Readers are referred to [19] for a detailed description of Nyström's method. See also [16] for the implementation of Nyström's method on integral equations generated by Helmholtz equation. Based on Kirsch and Monk's idea in [20] and [21], the exterior problem is solved by integral equation with radiation condition.

Define the space

$$W(\Omega_0) = \{u^s \in H_{\text{loc}}^1(\Omega_0) \mid \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - i k_0 u^s \right) = 0, \quad r = |x|\}. \quad (4.1)$$

Define the operators

$$G_i : H^{\frac{1}{2}}(\Gamma) \rightarrow H^1(\Omega), \quad (4.2)$$

$$G_e : H^{\frac{1}{2}}(\Gamma) \rightarrow W(\Omega_0) \quad (4.3)$$

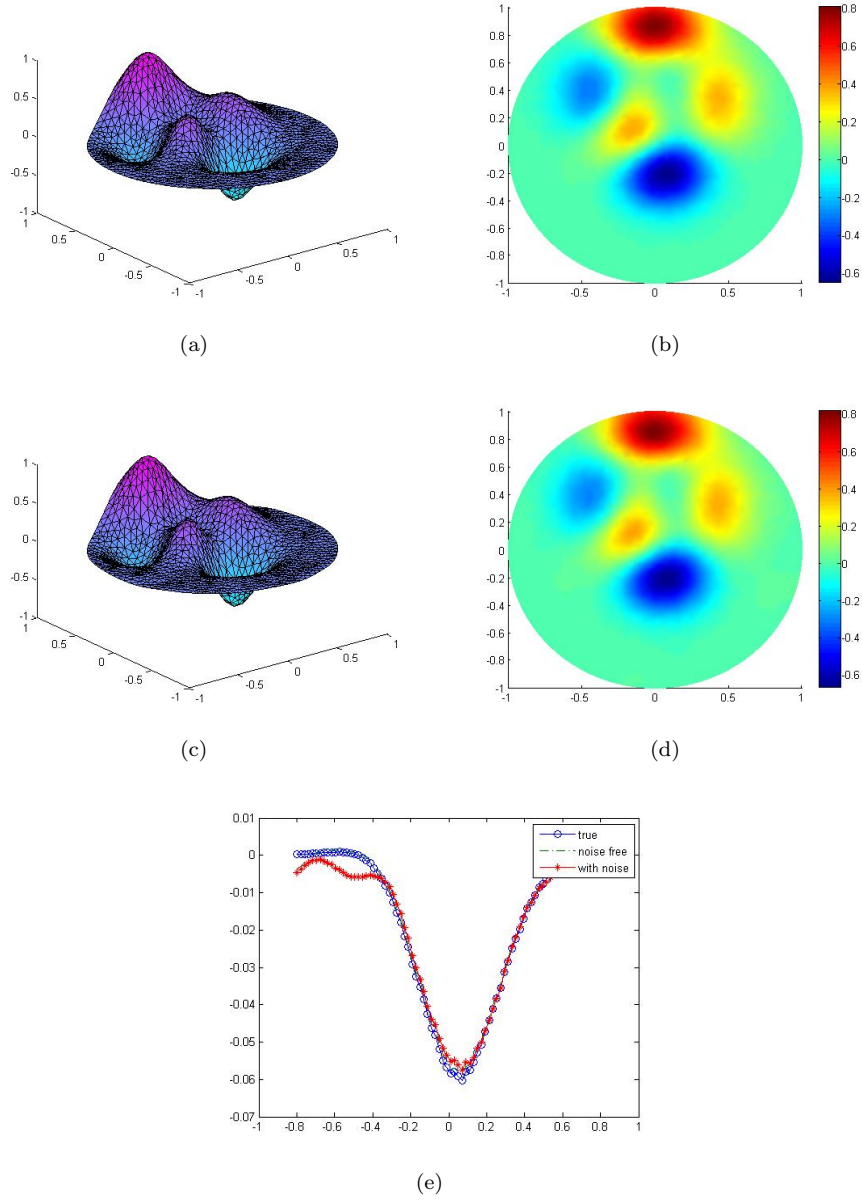


FIGURE 2. Example 1: (a,b): surface and contour views of the true scatterer function; (c,d): final reconstruction of the scatterer function with noisy data. (e): comparison of the true scatterer, final reconstruction without noise and final reconstruction with noisy data at cross section $y = -0.6$.

by the following boundary problems. Given $\lambda_\Gamma \in H^{\frac{1}{2}}(\Gamma)$, define $G_i \lambda_\Gamma = w$ where $w \in H^1(\Omega)$ is the weak solution of

$$\begin{aligned} \Delta w + k^2(1 + q_{\bar{k}}(k, x))w &= 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} + ikw &= \lambda_\Gamma & \text{on } \Gamma. \end{aligned} \quad (4.4)$$

Similarly define $G_e \lambda_\Gamma = w$ as the weak solution of

$$\begin{aligned} \Delta w + k^2 w &= 0 & \text{in } \Omega_0, \\ \frac{\partial w}{\partial n} + ikw &= \lambda_\Gamma & \text{on } \Gamma, \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial w}{\partial r} - ikw \right) &= 0, \quad r = |x|. \end{aligned} \quad (4.5)$$

To ensure continuity of solution of the forward problem across Γ , it suffices to choose $\lambda_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ such that

$$G_i \lambda_\Gamma + G_i \left(\frac{\partial u^i}{\partial n} + ik_0 u^i \right) = G_e \lambda_\Gamma + u^i \quad \text{on } \Gamma. \quad (4.6)$$

The function λ_Γ is approximated by trigonometric polynomials of order N . Represent Γ by $x(t) = (r_\Gamma \cos \theta, r_\Gamma \sin \theta)$, $0 \leq \theta \leq 2\pi$. Write $\lambda_\Gamma = \sum_{n=-N}^{N-1} a_n \exp(in\theta_1)$. Thus for (4.6), $2N + 1$ finite element problems need to be solved on the left hand side, and since Γ is a circle, we can compute $G_e \lambda_\Gamma$ explicitly as a finite linear combination of Hankel functions:

$$G_e \lambda_\Gamma = \frac{1}{k_0} \sum_{n=-N}^{N-1} \frac{a_n \exp(in\theta)}{(H_n^{(1)})'(k_0 r_\Gamma) + iH_n^{(1)}(k_0 r_\Gamma)} H_n^{(1)}(k_0 r) \quad (4.7)$$

As for the adjoint problem, the continuity conditions are:

$$G_i \lambda_\Gamma + G_i((\overline{u - \tilde{u}})k_b^2) = G_e \lambda_\Gamma - G_e((\overline{u - \tilde{u}})k^2) \quad \text{on } \Gamma. \quad (4.8)$$

5. NUMERICAL EXPERIMENTS

In the following, to illustrate the performance of the algorithm, two numerical examples are presented for reconstructing the scatterer of the Helmholtz equation in

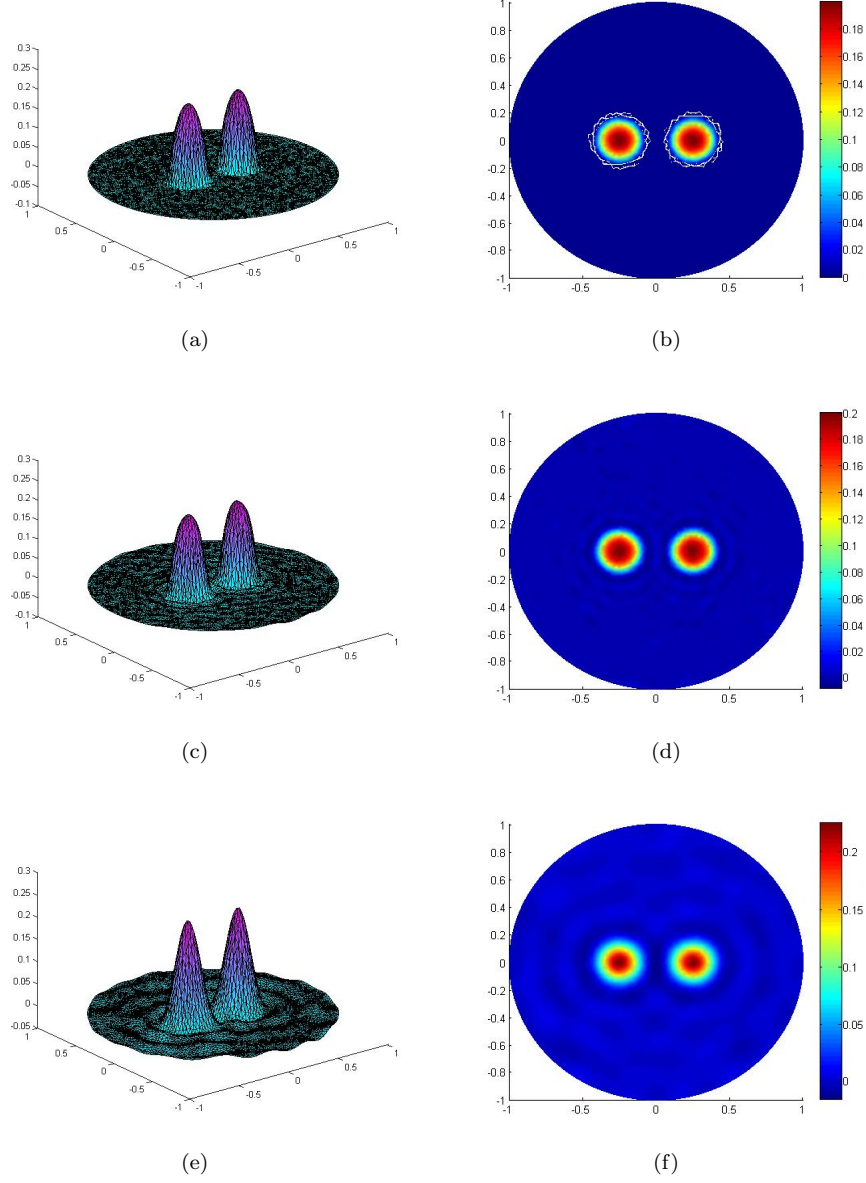


FIGURE 3. Example 2: (a,b): surface and contour views of the true scatterer function; (c,d): Final reconstruction of the scatterer function with noise-free data; (e,f): final reconstruction of the scatterer function with noisy data.

two dimensions. We performed four reconstructions, each example with noise-free and noisy data that contains 2% multiplicative noise. The scattered field takes the

form $u^s|_\Gamma := (1 + 0.02 \text{ rand})u^s|_\Gamma$, where rand gives uniformly distributed random numbers in $[-1, 1]$.

Example 5.1. Let

$$\begin{aligned} \sigma(x, y) = & 0.3(1 - x)^2 \exp(-x^2 - (y + 1)^2) \\ & - \left(\frac{x}{5} - x^3 - y^5\right) \exp(-(x^2 + y^2)) - \frac{1}{30} \exp(-(x + 1)^2 - y^2). \end{aligned}$$

Define $q(k_0, x) = i\sigma(3x, 3y - 1)/k_0$ in Ω . See figure 2 for the surface plot of the scatterer function in the domain $|x| < 1$, the result of the final reconstructions with noise-free and noisy data using the wavenumber $k_0 = 10.1$, which has relative error 1.64% and 2.49%, respectively.

Example 5.2. In this example, we performed the reconstruction with scatterer function

$$q(k_0, x) = \begin{cases} i\frac{0.2}{k_0} & (x + 0.25)^2 + y^2 < 0.2^2 \quad \text{in } \Omega, \\ i\frac{0.2}{k_0} & (x - 0.25)^2 + y^2 < 0.2^2 \quad \text{in } \Omega, \\ 0 & \text{elsewhere in } \Omega. \end{cases}$$

Using the wavenumber $k_0 = 12.1$, the final reconstructions with noise-free and noisy data have relative errors 5.23% and 16.05%. The functions and reconstructions are shown in figure 3.

6. CONCLUSIONS

In this paper we have presented a continuation method on an inverse scattering problem in dispersive media using multi-frequency data. The numerical results showed the efficiency and robustness of the algorithm. Our future project is to vary the real part of the dielectric constant, in which case both real and imaginary part of the scatterer need to be reconstructed. Another direction is to consider the three dimensional case.

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